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Turbulence in a Rarefied Plasma

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Abstract

A theory is developed on the damping process in a turbulent plasma (nonlinear Landau damping). The Vlasov-Poisson equations, describing the turbulent fluctuations, generate a chain of equations for their correlations. If the fourth order correlations are degenerated into lower orders, the chain becomes closed. The result shows that the decay of the electrostatic wave energy is governed by a linear damping mechanism, i.e. the linear Landau damping of waves which is independent of the amplitudes, and a turbulent damping depending on the amplitudes.

author

We assume the background distribution function $f(v)$ to be a given function of v ; then the fluctuation $F(t,\underset{\sim}{x},\underset{\sim}{v})$ satisfies the following equation, derived from the Vlasov equation:

$$\frac{\partial F}{\partial t} + \underset{\sim}{v} \frac{\partial F}{\partial \underset{\sim}{x}} + \frac{e}{m} \underset{\sim}{E} \frac{\partial f}{\partial \underset{\sim}{v}} = -\frac{e}{m} \frac{\partial}{\partial \underset{\sim}{v}} (\underset{\sim}{E} F) ,$$

or, in the form of its Fourier transform in wave-number $\underset{\sim}{k}$,

$$\frac{\partial F(t,\underset{\sim}{k}, \underset{\sim}{v})}{\partial t} + i\underset{\sim}{k}\,\underset{\sim}{v}\, F = \frac{ie}{m} \underset{\sim}{k}\, \Phi(t,\underset{\sim}{k}) \frac{\partial f}{\partial \underset{\sim}{v}} + \frac{e}{m} \frac{\partial}{\partial \underset{\sim}{v}} \int d\underset{\sim}{k}' i\underset{\sim}{k}' \Phi(t,k')\, F(t,\underset{\sim}{k} - \underset{\sim}{k}',\underset{\sim}{v}) \tag{1}$$

where $\underset{\sim}{E}$ is the self-consistent electric field, and Φ is its corresponding potential, defined by :

$$\frac{\partial \Phi}{\partial \underset{\sim}{x}} = - \underset{\sim}{E}\,(t,\underset{\sim}{x})$$

In order to distinguish between the various species of particles, an index j may be added if necessary. Further the Poisson equation is

$$\Phi(t,\underset{\sim}{k}) = \sum_j \frac{4\pi n_j e_j}{k^2} \int d\underset{\sim}{v}\, F^j(t,\underset{\sim}{k},\underset{\sim}{v}), \tag{2}$$

where n_j and e_j are respectively the number density and the charge for the species j.

Upon integration of (1) and use of (2), we obtain the rate of change of the electric correlation, as given by the following equation:

$$\frac{\partial}{\partial t} \overline{\Phi(t,\underset{\sim}{k})\Phi(t,-\underset{\sim}{k})} = \sum_j \frac{\omega_j^2}{k^2} \int_{-\infty}^{\infty} d\underset{\sim}{v}\, \underset{\sim}{k} \cdot \underset{\sim}{v}\, \underset{\sim}{k} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} \int_{t'}^{t} dt_1 e^{-i\underset{\sim}{k}\,\underset{\sim}{v}\,(t-t_1)}\, \overline{\Phi(t_1,\underset{\sim}{k})\ \Phi(t,-\underset{\sim}{k})}$$

$$+ \sum_j \frac{\omega_j^2}{k^2} \int_{-\infty}^{\infty} d\underset{\sim}{v}\, \underset{\sim}{k} \cdot \underset{\sim}{v} \int d\underset{\sim}{k}'\underset{\sim}{k}' \cdot \frac{\partial}{\partial \underset{\sim}{v}} \int_{t'}^{t} dt_1\, e^{-i\underset{\sim}{k}\,\underset{\sim}{v}(t-t_1)}$$

$$\times\ \overline{\Phi(t_1,\underset{\sim}{k})\ \Phi(t,-\underset{\sim}{k})\ F(t_1,\underset{\sim}{k}-\underset{\sim}{k}',\underset{\sim}{v})} + c.c\ . \tag{3}$$

where t' is the initial time which can be set equal to zero. Since t is taken to be large, the contribution from the initial distribution function $F(t')$ is neglected.

For a stationary and homogeneous turbulence, we can introduce the definitions:

$$I_{\Omega} = k^2 \, \overline{\Phi(\omega,\underset{\sim}{k}) \; \Phi(-\omega,-\underset{\sim}{k})}$$

$$Q_{\Omega-\Omega',\Omega} = \overline{\Phi(-\omega,-\underset{\sim}{k}) \; \Phi(\omega',\underset{\sim}{k}') \; F(\omega-\omega',\underset{\sim}{k}-\underset{\sim}{k}')}$$

where Ω represents the pair of variables $(\omega\,,\,\underset{\sim}{k})$. We note that the two integrals entering in (3) are simply the Fourier transforms of I and Q respectively, and hence (3) becomes after such a transform,

$$\frac{\partial I_{\Omega}}{\partial t} = 2\pi \sum_j \frac{\omega_j^2}{k^2} \int d\underset{\sim}{v}\, \underset{\sim}{k}\cdot\underset{\sim}{v}\, \delta(\omega-\underset{\sim}{k}\,\underset{\sim}{v})\, \underset{\sim}{k}\cdot\frac{\partial f}{\partial \underset{\sim}{v}}\, I_{\Omega}$$

$$+ 2\pi \sum_j \frac{\omega_j^2}{k^2} \int d\underset{\sim}{v}\, \underset{\sim}{k}\cdot\underset{\sim}{v}\, \delta(\omega-\underset{\sim}{k}\,\underset{\sim}{v}) \int d\Omega'\, \underset{\sim}{k}'\cdot \frac{\partial Q_{\Omega-\Omega',\Omega}}{\partial \underset{\sim}{v}} \tag{4}$$

As shown in Appendix (A8), we have:

$$Q_{\Omega-\Omega',\Omega} = \sum_j \underset{\sim}{G}^{ij}_{\Omega-\Omega'} \left[\frac{\underset{\approx}{k}'}{k'}\, \tilde{\mu}_{\Omega} + \frac{\underset{\approx}{k}}{k}\, \tilde{\mu}_{\Omega'} \right] \frac{I_{\Omega'}}{kk'}\, I_{\Omega} \tag{5}$$

with $\underset{\sim}{G}$, $\widetilde{\mu}$ defined there. For a given amplitude I_{Ω} , the function $\widetilde{\mu}$ is determined by the integral equation (A7). Upon substitution of (5), we can rewrite (4) as follows:

$$\frac{\partial I_{\Omega}}{\partial t} = 2\,\widetilde{\gamma}\, I_{\Omega} \tag{6}$$

where

$$\widetilde{\gamma} = \gamma + \gamma_t \tag{7a}$$

$$\gamma = \pi \sum_j \frac{\omega_j^2}{k^2} \int d\underset{\sim}{v}\; \underset{\sim}{k} \cdot \underset{\sim}{v}\; \delta(\omega - \underset{\sim}{k}\,\underset{\sim}{v})\, \underset{\sim}{k} \cdot \frac{\partial f^j}{\partial \underset{\sim}{v}} \tag{7b}$$

$$\gamma_t = \pi \sum_j \frac{\omega_j^2}{k^2} \int d\underset{\sim}{v}\; \underset{\sim}{k} \cdot \underset{\sim}{v}\; \delta(\omega - \underset{\sim}{k}\,\underset{\sim}{v})\, \underset{\sim}{k} \cdot \frac{\partial f_t^j}{\partial \underset{\sim}{v}} \tag{7c}$$

$$\underset{\sim}{k} \cdot f_t = \sum_{\ell} \int d\,\Omega'\, \underset{\sim}{k}' \cdot G^{j\ell}_{\Omega - \Omega'} \left[\frac{\underset{\sim}{k}'}{k'}\, \widetilde{\mu}_{\Omega} + \frac{\underset{\sim}{k}}{k}\, \widetilde{\mu}_{\Omega'} \right] \frac{I_{\Omega'}}{kk'} \tag{7d}$$

We conclude that the expression (7a) for the total damping $\widetilde{\gamma}$ consists of a laminar damping (7b), i.e. linear Landau damping, and a turbulent damping (7c), which depends on the amplitudes I, as involved in (7d). We note that the present method of the derivation of the decay of the electric energy gives the damping coefficient γ in a simple and direct manner, without involving the eigen value calculations.

APPENDIX A. CORRELATION FUNCTIONS IN A TURBULENT PLASMA

Introducing a Fourier transform for the disturbance of the distribution function,

$$F(t, \underset{\sim}{x}, \underset{\sim}{v}) = \int d\Omega F_{\Omega}(v)\, e^{-i(\omega t - \underset{\sim}{k}\,\underset{\sim}{x})}$$

in (1), where Ω is the pair $(\omega, \underset{\sim}{k})$, $d\,\Omega = d\omega\, d\underset{\sim}{k}$, and $\delta(\Omega) = \delta(\omega)\,\delta(\underset{\sim}{k})$, we obtain:

$$F_{\Omega} = \underset{\sim}{g}_{\Omega} \{\underset{\sim}{k}\,\Phi_{\Omega} f + \int d\Omega'\, \underset{\sim}{k}'\Phi_{\Omega'} F_{\Omega - \Omega'}\} \tag{A1}$$

with the operator

$$\underset{\sim}{g}_{\Omega}(\underset{\sim}{v}) = -\frac{e}{m}\frac{1}{\omega - \underset{\sim}{k}\,\underset{\sim}{v}}\frac{\partial}{\partial \underset{\sim}{v}}$$

We shall treat the turbulence as stationary and homogeneous, and define

$$\overline{\Phi_{\Omega'}\Phi^{*}_{\Omega}} = \frac{1}{k^2}\, I_{\Omega}\,\delta(\Omega - \Omega')$$

$$\overline{F_{\Omega'}\Phi^{*}_{\Omega}} = P_{\Omega}\,\delta(\Omega - \Omega')$$

$$\overline{F_{\Omega'}\Phi_{\Omega - \Omega'}\Phi^{*}_{\Omega''}} = Q_{\Omega'\,\Omega}\,\delta(\Omega - \Omega'')$$

$$\overline{\Phi_{\Omega'}\,\Phi_{\Omega-\Omega'}\,\Phi^*_{\Omega''}} = q_{\Omega'\Omega}\,\delta(\Omega-\Omega'')$$

As a characteristic behavior of nonlinear problems, we expect to generate, from the system of equations (A1) and (2), a chain of equations with ever increasing orders of correlation functions. Such a chain is cut off by replacing the quadruple correlation function by products of double correlation functions, and becomes

$$I_{\Omega} = k^2 \sum_j \nu_j\, P^j_{\Omega} \tag{A2}$$

$$P_{\Omega} = \underset{\sim}{g}_{\Omega} \{ \frac{\underset{\sim}{k}}{k^2} f\, I_{\Omega} + \int d\Omega'(\underset{\sim}{k} - \underset{\sim}{k}')\, Q_{\Omega'\,\Omega} \} \tag{A3}$$

$$Q_{\Omega'\,\Omega} = \underset{\sim}{g}_{\Omega} \{ \underset{\sim}{k}' f\, q_{\Omega'\,\Omega} + [\frac{\underset{\sim}{k}' - \underset{\sim}{k}}{|\underset{\sim}{k}' - \underset{\sim}{k}|^2} I_{\Omega'-\Omega} P_{\Omega} + \frac{\underset{\sim}{k}}{k^2} I_{\Omega} P_{\Omega'-\Omega}]\} \tag{A4}$$

where ν_j is the operator

$$\nu_j = \frac{4\pi m_j e_j}{k^2} \int d\underset{\sim}{v}$$

We can eliminate q from (2) and (A4) to obtain:

$$Q^i_{\Omega'\Omega} = \sum_j \underset{\sim}{G}^{ij}_{\Omega'} [\frac{\underset{\sim}{k}' - \underset{\sim}{k}}{|\underset{\sim}{k}' - \underset{\sim}{k}|^2} I_{\Omega'-\Omega} P_{\Omega} + \frac{\underset{\sim}{k}}{k^2} I_{\Omega} P_{\Omega'-\Omega}] \tag{A5}$$

with the operator

$$\underset{\sim}{G}^{ij}_{\Omega} = \delta_{ij}\ \underset{\sim}{g}^{j}_{\Omega} + \frac{4\pi\ n_i e_i}{k\ \epsilon_{\Omega}}\ \mu^{i}_{\Omega} \int d\underset{\sim}{v}\ \underset{\sim}{g}^{j}_{\Omega}$$

Define

$$\mu_{\Omega} = \frac{\underset{\sim}{k}}{k}\ \underset{\sim}{g}_{\Omega}\ f$$

$$\epsilon_{\Omega} = 1 - \sum_{j} \frac{4\pi\ n_j e_j}{k} \int d\underset{\sim}{v}\ \mu^{j}_{\Omega}$$

We note that ϵ_{Ω} is the dielectric constant for a laminar plasma.

If we write P in the form:

$$P_{\Omega} = \frac{1}{k}\ \tilde{\mu}_{\Omega}\, I_{\Omega} \tag{A6}$$

where $\tilde{\mu}_{\Omega}$ depends on f and I_{Ω}, we obtain from (A3) and (A5) the following equation determining $\tilde{\mu}_{\Omega}$:

$$\tilde{\mu}^{i}_{\Omega} = \mu^{i}_{\Omega} + \vec{\underset{\sim}{g}}^{i}_{\Omega'} \cdot \int d\Omega\ \frac{\vec{\underset{\sim}{k} - \underset{\sim}{k}'}}{|\underset{\sim}{k} - \underset{\sim}{k}'|}\ I_{\Omega'-\Omega} \sum_{j} \underset{\sim}{G}^{ij}_{\Omega''} \cdot \left[\frac{\underset{\sim}{k} - \underset{\sim}{k}'}{|\underset{\sim}{k}' - \underset{\sim}{k}|}\ \tilde{\mu}^{j}_{\Omega} + \frac{\underset{\sim}{k}}{k}\ \tilde{\mu}^{j}_{\Omega'-\Omega} \right] \tag{A7}$$

Here $\tilde{\mu}_{\Omega}$ gives the effective dielectric coefficient for a turbulent plasma in the form:

$$\tilde{\epsilon}_{\Omega} = 1 - \sum_{j} \frac{4\pi\, n_j e_j}{k} \int d\underset{\sim}{v}\ \tilde{\mu}^{j}_{\Omega}(\underset{\sim}{v})$$

As we could expect and verify by means of (A2) and (A6), the dispersion relation is given by

$$\tilde{\epsilon}_{\Omega} = 0$$

We note that we can write (A5) in an alternate form in terms of $\tilde{\mu}$ as follows:

$$Q^{i}_{\Omega'\Omega} = \sum_{j} \tilde{G}^{ij}_{\Omega'} \left[\frac{\underset{\sim}{k}'-\underset{\sim}{k}}{|\underset{\sim}{k}'-\underset{\sim}{k}|} \tilde{\mu}_{\Omega} + \frac{\underset{\sim}{k}}{k} \tilde{\mu}_{\Omega'-\Omega} \right] \frac{I_{\Omega} I_{\Omega'-\Omega}}{k\,|\underset{\sim}{k}'-\underset{\sim}{k}|} \qquad \text{(A8)}$$

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